PMM U.S.S.R.,Vol.51,No.5,pp.687-691,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

FRONTS OF STATIONARY NON-LINEAR WAVES IN MEDIA WITH A MEMORY AND FACTORIZATION THEOREMS*

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Continuous and discontinuous one-dimensional waves of stationary profile are studied in non-linearly hereditary rods, where the heredity kernel is not assumed to be regular (i.e., the existence of a singularity is allowed for in the kernel, an integrable singularity as $t \to +0$). Appropriate near-front asymptotic forms are found. As is well-known /1/, strong discontinuities cannot propagate in hereditary media with singular kernels in a linear situation. Consequently, the very possibility of the existence of solutions with strong discontinuities in the non-linear singular case is not totally trivial.

A hypothesis is formulated on the possibility of soliton-like solutions being propagated for the case when the heredity kernel satisfies certain additional constraints. The investigation of waves of stationary profile reduces to finding non-zero solutions of Volterra-type non-linear integral equations without right sides. Two new theorems are presented on the factorization of non-linear wave operators with a memory and a kindred theorem on the asymptotic factorization of the Klein-Fock-Gordon equation. These theorems enable the research results to be extended to a broad class of governing relations.

This paper continues the investigations in /2-6/. The waves of stationary profile /3/ corresponded to the case of an exponential heredity kernel, which enabled the problem on determining the shape of such a wave to be reduced to the solution of an ordinary differential equation. A more general situation is examined in /6/ where a number of results on waves of stationary profile are given and partially proved for regular kernels. The factorization theorems presented below are based on the results in /7, 8/.

1. Formulation of the problem.

We consider a non-linear hereditary elastic rod of density $\rho = \text{const}$ in which the strain ε and the stress σ are connected by the relationship

$$(1-\gamma R^{\smile})\,\varepsilon = A\sigma + \gamma B\sigma^2; \quad R^{\smile}\varepsilon \equiv \int_{-\infty}^{t} R(t-\tau)\,\varepsilon(\tau)\,d\tau \tag{1.1}$$

where $A > 0, 0 < \gamma \ll 1$. It is assumed here that the function $R(t)$ defined on the half-axis $t > 0$ is non-negative, decreases, and its integral over the whole half-axis is finite. As has been shown in /2, 7/, the stress wave propagating to the right in such a rod is described by the following one-wave equation:

$$(1-k\gamma\sigma)\,\partial_t\sigma + (1-{}^1/_2\gamma R^{\smile})\,\partial_x\sigma = 0; \quad k = -B/A, \quad x = y\sqrt{A\rho} \tag{1.2}$$

(y is the Lagrangian coordinate of points of the rod). The condition on the discontinuity in t, x coordinates has the form $1/U = 1 - k\gamma\,[\sigma^2]/(2\,[\sigma])$. Here U is the velocity of the front in the coordinates mentioned, and the square brackets denote taking the discontinuity at the front. To be specific, we assume $k > 0$ below. In the case $R^{\smile} \equiv 0$ this condition corresponds to reversal of the tension wave (moving to the right) during its motion.

2. Continuous stationary waves.

The question of whether waves exist in the one-dimensional medium under consideration, that propagate to the right in the unperturbed domain without changing shape is of interest. For waves that do not contain strong discontinuities the question formulated reduces to a question of whether solutions of the form $\sigma = f(t - x/c)$, $c > 0$ exist for (1.2), where $f(z) = 0$ for $z < 0$. As is well-known, perturbations of infinitesimal amplitude propagate over an unperturbed medium with an instantaneously elastic velocity (with unit velocity in t, x coordinates). Consequently, the required continuous solution of (1.2) should be sought in the form $\sigma = f(t - x)$. Substituting this expression into (1.2), integrating and redesignating the variable $z = t - x$ by t, we arrive at the selfsimilar

*Prikl.Matem.Mekhan.,51,5,871-876,1987

integral equation

$$kf^2(t) = \int_0^t R(t-\tau) f(\tau) d\tau \tag{2.1}$$

(the constant of integration turns out to equal zero since $f(t)=0$ for $t<0$). It is clear that if there is at least one non-zero solution $f(t)$ for Eq. (1.2), then an entire family of solutions of the form $f(t-t_0)$ will exist at once. A solution that does not vanish identically for arbitrarily small $t>0$ can obviously always be selected from this family.

Lemma. Let a bounded, monotonically increasing solution $f(t), t>0$ exist for (2.1) that is not identically zero in an arbitrarily small interval of the form $0<t<\delta$. Then the function $f(t)$ is continuous for $t \geqslant 0$, is infinitely differentiable for $t>0$, $f(0)=0$ and $f(t)>0$ for $t>0$. Moreover

$$f(t) \to f(\infty) = \frac{I}{k}, \quad t \to \infty; \quad I \equiv \int_0^\infty R(t) dt \tag{2.2}$$

$$\frac{1}{2k} \sqrt{R(t)} \int_0^t R(t) dt \leqslant f(t) \leqslant \frac{1}{k} \int_0^t R(t) dt \tag{2.3}$$

Only inequalities (2.3) actually require proof. To set them up it is sufficient to estimate the right side in (2.1) from above and below, respectively, in terms of $f(t) \int R(t) dt$ and $R(t) \int f(t) dt$. The infinite differentiability of $f(t)$ for $t>0$ results from the positivity of $f(t)$ for $t>0$ and the fact that the right-hand side in (2.1) has an order of smoothnesses exceeding the order of smoothness of the left side by one. The latter is evidently possible only under the condition of infinite differentiability of both sides in (2.1).

Theorem 1. The non-zero solution $f(t)$ of (2.1), mentioned in Lemma 1, exists.

We prove this theorem in two steps.

1°. Let

$$R(t) = Ct^{\alpha-1},\ 0<t<t_0;\ t_0>0,\ C>0,\ 0<\alpha\leqslant 1 \tag{2.4}$$

It is seen that the function

$$f(t) = C\Gamma(\alpha)\Gamma(\alpha+1)(k\Gamma(2\alpha+1))^{-1}t^{\alpha},\ 0\leqslant t<t_0 \tag{2.5}$$

is an exact solution of (2.1) with the kernel (2.4) in the half-interval $[0, t_0)$. We now construct the zeroth approximation to the solution in $[0,\infty)$ as follows. We consider $f_0(t)$ to have the form (2.5) for $0\leqslant t<t_0$ while we set $f_0(t) = I/k$ for $t\geqslant t_0$. We determine the next approximations recursively from the formula

$$kf_j^2(t) = \int_0^t R(t-\tau) f_{j-1}(\tau) d\tau \tag{2.6}$$

(here the positive value of the square root of the right side of (2.6) is taken in defining f_j in terms of f_{j-1}). It can be proved by induction that $f_j(t) \equiv f_0(t)$ in the half-interval $[0, t_0)$ under the selection made for the zeroth approximation, $f_j(t) \to I/k$ as $t\to\infty$ and all functions $f_j(t)$ are non-negative and grow monotonically. Moreover, by subtracting (2.6) for $j=n-1$ from an analogous equality for $j=n$ and representing the left side that is obtained in the form of a product $k(f_n+f_{n-1})(f_n-f_{n-1})$ it can be shown by induction that under the selection made for f_0 the inequalities $0\leqslant f_n(t)\leqslant f_{n-1}(t)\leqslant \ldots \leqslant f_0(t)$ will be valid for $t\geqslant 0$. Consequently, as $j\to\infty$ a non-zero limit exists for the functions $f_j(t)$ which we denote by $f(t)$ and which is the desired solution.

2°. Now let $R(t)$ be an arbitrary non-negative decreasing integrable function. We set $R_n(t) = R(1/n)$ for $0\leqslant t<1/n$ and $R_n(t) = R(t)$ for $t\geqslant 1/n$. It is clear that $R_1(t)\leqslant R_2(t)\leqslant\ldots\leqslant R_n(t)\leqslant\ldots\leqslant R(t)$, where all the $R_n(t)$ are decreasing non-negative integrable functions, each of which will satisfy conditions (2.4), respectively, with $C=R(1/n)$, $\alpha=1$, and $t_0=1/n$. Because of the preceding, (2.1) with kernel $R_n(t)$ has a monotonically increasing non-negative non-zero solution $f_{(n)}(t)$ for each n, for which the following limit relationship holds (see Lemma 1)

$$f_{(n)}(t) \to f_{(n)}(\infty) = \frac{1}{k}\int_0^\infty R_n(\tau) d\tau < \frac{1}{k}\int_0^\infty R(\tau) d\tau = \frac{I}{k}; \quad t\to\infty \tag{2.7}$$

Moreover, it can be proved by induction that $0<f_{(1)}(t)\leqslant\ldots\leqslant f_{(n)}(t)\leqslant\cdots$. But it follows from the monotonic growth of the function $f_{(n)}$ and from (2.7) that all the $f_{(n)}$ have the constant I/k as upper bound. Therefore, as $n\to\infty$ a limit exists for the sequence of function $f_{(n)}$ which is indeed the desired solution $f(t)$.

Remark 1°. An analogous theorem obviously also holds for the more general equation $\varphi(f) = R^{\smile} f$ where φ is an arbitrary monotonically increasing function of f having a zero of second order for $f = 0$.

2°. Let $R(t) = Ce^{-\alpha t} \sin(\omega t + \varphi_0)$ for $t \geqslant 0$, where $C > 0, \alpha \geqslant 0, \omega > 0, 0 \leqslant \varphi_0 < \pi$. Then (2.1) is reduced to the second-order differential equation

$$(f^2)'' + 2\alpha (f^2)' - C_1 \sin \varphi_0 \cdot f' - C_1 (\omega \cos \varphi_0 + \alpha \sin \varphi_0) f + (\alpha^2 + \omega^2) f^2 = 0$$

with initial conditions

$$f(0) = 0,\ f'(0) = {}^1/_2 C_1 \sin \varphi_0;\ C_1 = C/k$$

In particular, (2.1) has the following solution for $R(t) = C \sin \omega t$:

$$f(t) = C_2 (1 - \cos(\omega t/2));\ C_2 = 2C/(3\omega k)$$

Hypothesis. If the non-negativity of the kernel $R(t)$ is discarded and it is assumed that its integral along the half-axis $t > 0$ is zero, then (2.1) will have a soliton-like solution.

3. Discontinuous stationary waves. An equation analogous to (2.1) that describes discontinuous waves of stationary profile propagating in an unperturbed medium has the form

$$g(t)(g(t) - g(+0)) = A_0 \int_0^t R(t-\tau) g(\tau)\, d\tau; \quad A_0 \equiv \frac{1 - k\gamma g(+0)/2}{k} \tag{3.1}$$

It is here assumed that the small parameter γ is sufficiently small so that $A_0 > 0$. The variable t in (3.1) is considered to be non-negative. Let us note the non-standard self-consistent nature of (3.1): the quantity $g(+0)$, the "initial condition" for the function $g(t)$, enters directly into (3.1). The stress wave $\sigma = g(t - x/c)$, $c = (1 - k\gamma g(+0)/2)^{-1}$ corresponds to the solution $g(t)$ of (3.1). Since $k > 0$ by assumption, it follows from the stability condition for a shock wave that there should be $g(+0) > 0$. Later, for simplicity we shall write $g(0)$ instead of $g(+0)$.

Lemma 2. For a certain $g(0) > 0$, let a bounded monotonically increasing solution $g(t)$, $t \geqslant 0$ exist for (3.1). Then

$$g(t) \to g(\infty) = g(0) + A_0 I,\ t \to \infty \tag{3.2}$$

$$\frac{g(0)}{2} + \left(\frac{g^2(0)}{4} + A_0 g(0) \int_0^t R(t)\, dt\right)^{1/2} \leqslant g(t) \leqslant g(0) + A_0 \int_0^t R(t)\, dt \tag{3.3}$$

where the function $g(t)$ is continuous for $t \geqslant 0$ and infinitely differentiable for $t > 0$.

Proof. The relationship (3.2) follows at once from the finiteness of the integral of the kernel $R(t)$ and from (3.1). Inequalities (3.3) can be obtained by estimating the right side in (3.1) from above and below, respectively, in terms of $A_0 g(t) \int R(t)\, dt$ and $A_0 g(0) \int R(t)\, dt$ (integration is between 0 and t).

Theorem 2. For each value $g(0) > 0$ a bounded monotonically increasing solution $g(t)$, $t \geqslant 0$ exists for (3.1).

The proof of this theorem (like the proof of Theorem 1) is performed in two steps.

1°. As the zeroth approximation to the solution we take an arbitrary continuous monotonically increasing function $g_0(t)$ such that $g_0(0) = g(0)$ and $g_0(\infty) = g(0) + I/k$. We then determine the successive approximations by means of the formula

$$g_n(t)(g_n(t) - g(0)) = A_0 \int_0^t R(t-\tau) g_{n-1}(\tau)\, d\tau; \quad t \geqslant 0 \tag{3.4}$$

(here the larger of the two roots of the quadratic equation is selected to define g_n in terms of g_{n-1}). The convergence of the approximations constructed in the small segment $0 \leqslant t \leqslant \delta$ to a certain montonically increasing function $\varphi(t)$ can be proved. For this it is sufficient to set up the inequality $\max|g_m - g_{m-1}| \leqslant \varkappa \max|q_{m-1} - q_{m-2}|$, $0 < \varkappa < 1;\ m = 1, 2, \ldots$ for $0 \leqslant t \leqslant \delta$. But this inequality can be proved by subtracting (3.4) for $n = m - 1$ from (3.4) for $n = m$ and taking the upper limit of the integral component.

2°. We will now establish the existence of a solution on the whole semi-axis. We use the successive approximation scheme (3.4) again, however we take the function $g_0(t) = \varphi(t)$ for $0 \leqslant t \leqslant \delta$ and $g_0(t) = g(0) + I/k$ for $t > \delta$ as the zeroth approximation this time. Convergence of the successive approximations determined by such a method on the whole semi-axis $t \geqslant 0$ follows from the fact that all the functions $g_n(t)$ increase monotonically, have a common constant constant as upper bound, satisfy the inequalities $0 < g_n(t) \leqslant g_{n-1}(t) \leqslant \ldots \leqslant g_0(t)$ proved by induction, and $g_n(t) \equiv \varphi(t)$ holds in the segment $0 \leqslant t \leqslant \delta$ for all n. The limit function $g(t)$ is indeed the desired solution of (3.4). The theorem is proved.

4. Stabilizing waves. It is clear from (3.1) and (3.2) that $g(\infty) > f(\infty) = I/k$. Consequently, the following qualitative deductions can be made about the behaviour of the solutions of (1.2) at large times for initial and boundary conditions given below (Θ is the Heaviside function)

$$x = 0,\ \sigma = \sigma_0 \Theta(t)\ (\sigma_0 > 0);\ t \leqslant 0,\ x > 0,\ \sigma(t, x) = 0 \tag{4.1}$$

1°. If $\sigma_0 > I/k$, then as $t \to \infty$ the solution of problem (1.2), (1.4) tends to a discontinuous wave of stationary profile propagating over an unperturbed medium: $\sigma = g(t - x/c + \text{const})$. The front velocity c and the magnitude of the discontinuity on the front $g(0)$ are determined here from the equalities $1/c = 1 - k\gamma g(0)/2$ and $\sigma_0 = g(\infty) \equiv I/k + g(0)(1 - \gamma I/2)$. Therefore, the non-linear effect of overturning turns out to be stronger here than the smoothing because of relaxation.

2°. If $\sigma_0 = I/k$, then as $t \to \infty$ the solution of problem (1.2), (1.4) tends to a continuous stationary wave of the form $\sigma = f(t - x + \text{const})$. It can be said that the non-linear and relaxation effects are mutually equilibrated in this case.

3°. If $0 < \sigma_0 < I/k$, as follows from the results obtained above, no solution exists for any of the selfsimilar Eqs.(2.1) and (3.1), that tends to σ_0 at infinity. It can be shown, however, that even in this case the wave profile is stabilized, but the wave profile being propagated over a perturbed medium turns out to be the limit.

5. Factorization theorems. Underlying the theory was the one-wave Eq.(2.1) containing a quadratic non-linearity. This equation was obtained in /2, 7/ as the asymptotic form as $\gamma \to 0$. It follows from the theorems presented below that a) a governing relationship exists for which (2.1) is not asymptotic but exact; b) Eq.(2.1) allows of natural extension to the case of a non-linearity of general form. It can be shown that all the results of Sects.2-4 are carried over to the case of a non-linearity of general form.

Theorem 3. Let the governing relationship

$$\varepsilon(t) = \int_{-\infty}^{t} (1 + R^{\smile}) \sqrt{a'(\sigma)}\, (1 + R^{\smile}) \sqrt{a'(\sigma)}\, \sigma_t'\, dt$$

hold.

Then the dynamic equation for the stress

$$\partial_t^2 \int_{-\infty}^{t} (1 + R^{\smile}) \sqrt{a'(\sigma)}\, (1 + R^{\smile}) \sqrt{a'(\sigma)}\, \sigma_t'\, dt - \frac{1}{\rho} \partial_y^2 \sigma = 0$$

is factorized exactly as follows

$$\left\{\partial_t (1 + R^{\smile}) \sqrt{a'(\sigma)} \mp \frac{1}{\sqrt{\rho}} \partial_y\right\} \left\{(1 + R^{\smile}) \sqrt{a'(\sigma)}\, \partial_t \pm \frac{1}{\sqrt{\rho}} \partial_y\right\} \sigma = 0 \tag{5.1}$$

(the upper or lower signs are selected simultaneously). The one-wave equations corresponding to the factorization (5.1) have the form

$$(1 + R^{\smile}) \sqrt{a'(\sigma)}\, \partial_t \sigma \pm \frac{1}{\sqrt{\rho}} \partial_y \sigma = 0$$

Another approach to this problem is described in /5/.

The governing relationship used in Theorem 3 is constructed in a rather complicated matter. Still another simpler governing relationship of hereditary type (also containing a non-linearity of general form) can be proposed for which factorization of the appropriate wave operator also turns out to be possible. However, unlike (5.1) this factorization will not be exact but asymptotic with a uniformly small residual.

Theorem 4. Let the following relationship hold

$$\varepsilon(t) = a(\sigma(t)) + \gamma \int_{-\infty}^{t} d(\sigma(\tau))\, d\tau, \quad 0 < \gamma \ll 1$$

Then the appropriate dynamic equation for the stress

$$\partial_t^2 a(\sigma) + \gamma \partial_t b(\sigma) - \rho^{-1} \partial_y^2 \sigma = 0$$

can be factorized asymptotically in the following manner:

$$\left\{\partial_t \sqrt{a'(\sigma)} + \frac{\gamma}{2}\beta'(\sigma) \mp \frac{1}{\sqrt{\rho}}\partial_y\right\}\left\{\sqrt{a'(\sigma)}\,\partial_t\sigma + \frac{\gamma}{2}\beta(\sigma) \pm \frac{1}{\sqrt{\rho}}\partial_y\sigma\right\} = O(\gamma^2) \tag{5.2}$$

$$\beta(\sigma) \equiv (a'(\sigma))^{-1/4}\int_0^\sigma (a'(\eta))^{-1/4}\, b'(\eta)\, d\eta$$

where the quantity $O(\gamma^2)$ is uniformly small for a bounded σ and vanishes for $\sigma = 0$. The one-wave equations corresponding to the factorization (5.2) have the form

$$\sqrt{a'(\sigma)}\,\partial_t\sigma + \frac{\gamma}{2}\beta(\sigma) \pm \frac{1}{\sqrt{\rho}}\partial_y\sigma = 0$$

For completeness, we present one more factorization theorem similar to Theorems 3 and 4.

Theorem 5. the Klein-Fock-Gordon equation

$$\partial_t^2 w - \partial_y^2 w + \gamma g(w) = 0,\ 0 < \gamma \ll 1$$

cam be factorized asymptotically as follows

$$\{\partial_t + {}^1/_2\gamma 1^{\smile} g'(w) \mp \partial_y\}\{\partial_t w + {}^1/_2\gamma 1^{\smile} g(w) \pm \partial_y w\} = O(\gamma^2) \tag{5.3}$$

Here $1^{\smile}$ is the integration operator with respect to dt from $-\infty$ to t, while the quantity $O(\gamma^2)$ is uniformly small for bounded w and vanishes for $w = 0$. The one-wave equations corresponding to the factorization (5.3) have the form

$$\partial_t w + \frac{\gamma}{2}\int_{-\infty}^{t} g(w(\tau))\, d\tau \pm \partial_y w = 0$$

The proof of Theorems 3-5 reduce to multiplication of the operator brackets in (5.1)-(5.3). It is here necessary to take account of the order of the factor-operators (the operator on the right acts earlier).

The author is grateful to N.V. Zvolinskii for discussing the results.

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Translated by M.D.F.